

Complete markets and Arrow-Debreu assets

Pierre Chaigneau
`pierre.chaigneau@hec.ca`

July 6, 2011

States of the world

- ▶ A state of the world is defined as a given set of values for the states variables, which describe the state of the economy, commodity prices, unemployment, natural disasters, etc. Denote by S the set of possible (future) states.
- ▶ In a perfect world, an individual could specify his consumption in any possible state of the world. He would have a state-contingent consumption plan. Under which conditions can he do this (at least in principle) with financial assets?
- ▶ DEFINITION: Market are *complete* if any consumption plan can be spanned by an investment strategy.
- ▶ Markets are complete iff there exists S assets with linearly independent payoffs.

Arrow-Debreu assets

Contingent claims

- ▶ An implication of a world of complete markets is that there is no need to consider existing financial assets. Considering consumption and states of the world suffices!
- ▶ An *Arrow-Debreu asset* (also known as a *contingent claim*) pays off one unit of the consumption good (the numeraire) in a given state of the world s , and in this state only.
- ▶ If markets are complete, we can suppose that investors can trade contingent claims, because the existing assets span or synthesize all contingent claims.
 - ▶ This is not meant to be realistic. But if markets are complete, it is equivalent to work with Arrow Debreu assets or with existing financial assets, and the former is easier! All the results will be the same.
- ▶ Suppose that a riskfree asset does not exist. Are markets incomplete?

State prices

- ▶ The price of the Arrow-Debreu asset which pays off 1 in state s is called the *state price*, and is denoted by $q(s)$.
- ▶ PROPOSITION: In complete markets without arbitrage opportunities, the state price vector exists and it is unique.
 - ▶ Investors trade until MRS are equal across agents state by state (if not, there are opportunities for mutually beneficial trades).
 - ▶ In incomplete markets, several different state price vectors can be obtained in equilibrium (even in the absence of arbitrage opportunities).
- ▶ The investor portfolio is described by the number of Arrow-Debreu assets purchased, for any $s \in S$.
- ▶ We assume for simplicity that the investor does not have any other contingent income, so that his consumption plan is fully described by his portfolio.
- ▶ The number of state- s Arrow-Debreu assets purchased and consumption in state s , which we denote by $c(s)$, therefore coincide.

Arrow-Debreu equilibrium

Complete markets equilibrium

- ▶ Each agent $i \in I$ receives a state-contingent endowment $e^i(1), e^i(2), \dots, e^i(S)$.
- ▶ DEFINITION: An Arrow-Debreu equilibrium is a vector of state prices $q(s)_{s=\{1,\dots,S\}}$ and consumption plans $\{c(s)^i\}_{i=\{1,\dots,I\}, s=\{1,\dots,S\}}$ such that
 - ▶ The budget constraint of each agent is satisfied (in equilibrium):

$$c^i \in B^i(q(s)_{s=\{1,\dots,S\}}) = \left\{ c \in R_+^S \text{ s.t. } \sum_s q(s)c(s) = \sum_s q(s)e^i(s) \right\}$$

- ▶ Agents maximize their expected utility: $U(c^i) \geq U(c)$ for any $c \in B^i(q(s)_{s=\{1,\dots,S\}})$.
- ▶ All markets clear: $\sum_i c^i(s) = \sum_i e^i(s)$ for any s .

Asset pricing with contingent claims

- ▶ Consider any asset with next period payoff described by the vector x .
- ▶ This asset can be viewed as a package of contingent claims.
- ▶ Its price is equal to the sum of the prices of these contingent claims:

$$p(x) = \sum_s q(s)x(s)$$

Risk-neutral probabilities

- ▶ Denote by $\pi(s)$ the probability of the state of nature s , and by $c(s)$ consumption in this state. Then $\sum_s \pi(s) = 1$.
- ▶ Denote by $q(s)$ is the price at $t = 0$ of one unit of consumption in state s . It is the *state price*.
- ▶ Define

$$q \equiv \sum_s q(s) = \frac{1}{1 + r_f}$$

- ▶ Define the *risk-neutral probability* as

$$\pi^*(s) \equiv \frac{q(s)}{q} \tag{1}$$

Asset pricing with risk neutral probabilities

- ▶ Consider any asset with next period payoff described by the vector x .
- ▶ Let's price this asset using risk neutral probabilities instead of state prices:

$$\begin{aligned} p(x) &= \sum_s q(s)x(s) = \sum_s \pi^*(s)q x(s) \\ &= \sum_s \pi^*(s) \frac{x(s)}{1 + r_f} = \frac{E^*[x]}{1 + r_f} \end{aligned}$$

The link between the SDF, risk-neutral probabilities, and state prices

- ▶ The usual first-order conditions give the state prices:

$$q(s) = \beta \pi(s) \frac{u'(c(s))}{u'(c_0)} = \pi(s) m(s) \quad (2)$$

- ▶ The relation between risk-neutral probabilities and the SDF m :

$$\frac{\pi^*(s)}{(1 + r_f)} = \pi(s) m(s) \quad \pi^*(s) = \frac{m(s)}{E[m(s)]} \pi(s)$$

- ▶ Risk-neutral probabilities do not include time discounting, while the SDF does not include (physical) probabilities.
- ▶ The state price accounts for time discounting, physical probabilities, and risk aversion.
- ▶ $\frac{q(s)}{\pi(s)}$, which is equal to the SDF (see equation (2)), is the price of one unit of consumption in state s per unit of probability. It is also called a *state price density*, or a *price kernel*.

Risk-neutral probabilities and the state price density

- Rewrite the FOC in (2) as

$$\frac{u'(c(s))}{u'(c_0)} = \frac{1}{\beta} \frac{q(s)}{\pi(s)} \quad (3)$$

- Consumption in state s depends only on the state price density in that state, $\frac{q(s)}{\pi(s)}$.
- Under risk neutrality, the LHS is a constant. The RHS is therefore also a constant ($\frac{q(s)}{\pi(s)} = \text{cste}$): it follows from (1) that the risk neutral probability $\pi^*(s)$ is equal to the physical probability $\pi(s)$.
- It also follows from (2) that m is a constant (non-stochastic) under risk neutrality: the variation of m across states gives the value of risk.

Risk-neutral probabilities and the state price density

- Using (1), rewrite (3) as

$$\pi^*(s) = \beta(1 + r_f) \frac{u'(c(s))}{u'(c_0)} \pi(s) \quad (4)$$

$$\pi^*(s) = k u'(c(s)) \pi(s) \quad \sum_s \pi^*(s) = k \sum_s u'(c(s)) \pi(s) = 1$$

- Under risk aversion, the RHS of (4) is decreasing in consumption: given $\pi(s)$, $\pi^*(s)$ is decreasing in $c(s)$.
- If $c(s) > c(\tau)$, then $\frac{\pi^*(s)}{\pi(s)} < \frac{\pi^*(\tau)}{\pi(\tau)}$.
- Since the ratio $\frac{\pi^*}{\pi}$ is decreasing in c , and since π and π^* are probabilities, we know that any state s with sufficiently high consumption will be characterized by $\pi^*(s) < \pi(s)$ (underweight). Conversely, any state s with sufficiently low consumption will be characterized by $\pi^*(s) > \pi(s)$ (overweight).

Risk-neutral probabilities and the state price density

- ▶ Since marginal utility is decreasing under risk aversion, consumption is decreasing in the state price density, which is intuitive.
- ▶ In the aggregate, the sum of individual consumptions must be equal to the total resources available in the economy in any given state (aggregate wealth).
- ▶ What does it suggest on the relation between the state price density and aggregate wealth?

Consumption and state prices

Optimal risk sharing

- Normalize the utility function by setting $u'(c_0) = 1$ for notational convenience, differentiate both sides of (3) w.r.t. the state price density:

$$\frac{dc(s)}{d\frac{q(s)}{\pi(s)}} u''(c(s)) = \frac{1}{\beta} \quad (5)$$

- Combine with (3) to get

$$\frac{dc(s)}{d\frac{q(s)}{\pi(s)}} = - \left(A(c(s)) \frac{q(s)}{\pi(s)} \right)^{-1} < 0 \quad (6)$$

- The LHS is the sensitivity of consumption to the state price density, which is negative. This means that the demand for a given Arrow-Debreu security (or equivalently for consumption in a given state s) is decreasing in the state price density.

Consumption and state prices

Optimal risk sharing

- ▶ The higher the coefficient of absolute risk aversion A , the less consumption is sensitive to the state price density.
 - ▶ An infinitely risk averse agent has a constant c , with the same level of consumption in any state of the world, regardless of the state prices. Smooth consumption at all costs!
- ▶ Tradeoff between smooth consumption and expected consumption. State prices are such that economic agents share risk optimally.
- ▶ $\frac{dc(s)}{d\frac{q(s)}{\pi(s)}}$ is increasing in the state price if the agent is prudent. In this case, the optimal consumption plan $c\left(\frac{q(s)}{\pi(s)}\right)$ is decreasing and convex.

The mutuality principle

Only aggregate risk matters

- ▶ Since all agents face the same prices, equation (3) implies that, for any two investors i and j with the same beliefs:

$$\beta \frac{u'(c^i(s))}{u'(c_0^i)} = \beta \frac{u'(c^j(s))}{u'(c_0^j)} \quad \text{for any } s \quad (7)$$

- ▶ MRS between current and any future consumption are equalized across investors in equilibrium. NB: investors can have different preferences, different levels of wealth, etc.!
- ▶ Investors are insured against idiosyncratic shocks in equilibrium (pooling of risks). Only aggregate/systematic shocks have an effect on individual consumption.
 - ▶ Systematic risk should NOT be pooled! Ex: AIG.
- ▶ What happens if one investor is risk neutral?

The risk sharing rule

How risk is allocated in an equilibrium with complete markets

- ▶ We already know that the marginal rate of substitution between consumption in a given state at $t = 1$ and consumption at $t = 0$ is the same for different agents (here i and j):

$$\beta \frac{u'(c_{t=1}^i(s=1))}{u'(c_{t=0}^i)} = \beta \frac{u'(c_{t=1}^j(s=1))}{u'(c_{t=0}^j)}$$

$$\beta \frac{u'(c_{t=1}^i(s=2))}{u'(c_{t=0}^i)} = \beta \frac{u'(c_{t=1}^j(s=2))}{u'(c_{t=0}^j)}$$

- ▶ Therefore

$$\frac{u'(c_{t=1}^i(s=2))}{u'(c_{t=1}^i(s=1))} = \frac{u'(c_{t=1}^j(s=2))}{u'(c_{t=1}^j(s=1))}$$

- ▶ The MRS between consumption in any two future states (here $s = 1$ and $s = 2$) will also be the same for different agents.

The representative agent approach

How to derive the SDF

- ▶ Like Arrow-Debreu assets, the representative agent is a useful abstraction.
 - ▶ Like no one believes in father Christmas, no one believes that the representative agent actually exists.
- ▶ Under complete markets, we only need consider one agent with the same preferences as the other agents¹ who consumes aggregate consumption (which is also aggregate production).
- ▶ Thus, we only need to know individual preferences in the face of risk and macroeconomic fundamentals (GDP) to derive the marginal rate of substitution of the representative agent for any two states.
- ▶ But knowing the MRS between consumption in each future state of the world at $t = 1$ and consumption at $t = 0$ gives us the Stochastic Discount Factor!

¹I'm simplifying here. Individuals do not need to have the same preferences for the representative agent approach to be valid.

Takeaway

- ▶ With complete markets, idiosyncratic risk does not matter, in the sense that individuals pool idiosyncratic risk in equilibrium, so that their consumptions are independent of their respective exposures to idiosyncratic risks. (cf. equation (7)).
- ▶ Only aggregate risk, i.e., fluctuations in aggregate endowment or aggregate consumption, matter.
- ▶ The optimal allocation of this aggregate risk depends on the risk aversion of the different agents: less risk averse agents bear a larger part of the risk (cf. equation (6)).
 - ▶ Agents face a tradeoff between maximizing their expected consumption, or smoothing their consumption across states of nature. The more risk averse they are, the more they lean toward the latter.
- ▶ In equilibrium, all aggregate risk must be borne! State prices adjust so that, in any state, the sum of individual consumptions (aggregate consumption) is equal to the sum of individual endowments (aggregate endowment).

Incomplete markets

- ▶ What if markets are incomplete? Two consequences:
 - ▶ Economic agents cannot conduct all mutually beneficial trades.
 - ▶ MRS are therefore not always equal across agents, which indicates that risk sharing (the allocation of risks, i.e. variations in consumption across states of the world) is suboptimal.
- ▶ We lose the tractability and simplicity of the complete markets framework:
 - ▶ We cannot use Arrow-Debreu assets.
 - ▶ We cannot use the representative agent approach.
- ▶ We need to solve each investor's problem individually, with the existing financial assets.
- ▶ The resulting equilibrium is not Pareto efficient, but it is constrained-Pareto efficient relative to the existing financial assets.

Incomplete markets

- ▶ Opportunities for efficiency improvements. Financial innovation: create financial assets to complete markets.

“Risk sharing has been used primarily for certain narrow kinds of insurable risks, such as stock market crashes or hurricanes, or for managing the risks of conventional investments, such as diversifying investment portfolios or hedging commodity risks. (...) Finance has substantially neglected the protection of our ordinary riches, our careers, our homes”

Robert Shiller, *The New Financial Order*

- ▶ In other words, financial assets have mainly been devised to trade systematic risks, but it is still hard to trade (and pool) idiosyncratic risks.
 - ▶ Efficient risk sharing already exists (almost) for some idiosyncratic risks: fire insurance, car insurance, medical insurance, even weather insurance!
 - ▶ Some risks have yet to be efficiently shared with the creation of appropriate financial instruments. For example: real estate valuation in a neighborhood, earnings prospects as a violinist,

Application: fixed income pricing

- ▶ Consider a zero coupon bond with a repayment of 100 in t periods in the absence of default. Default occurs with probability π , and triggers a repayment equal to $100 \times \text{recovery ratio}$ at maturity. The final payment is called the nominal value or face value of the bond.
- ▶ The bond price is equivalently given by:

$$P_0 = \frac{100}{(1 + r_f + \text{spread})^t} \quad (8)$$

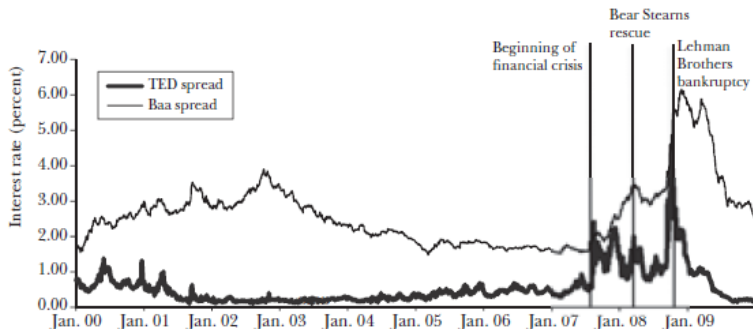
$$P_0 = \frac{(1 - \pi)100 + \pi(100 \times \text{recovery ratio})}{(1 + r_f + \text{risk premium})^t} \quad (9)$$

$$P_0 = \frac{(1 - \pi^*)100 + \pi^*(100 \times \text{recovery ratio})}{(1 + r_f)^t} \quad (10)$$

Application: fixed income pricing

Credit spreads and economic conditions (Source: Mishkin, JEP 2011)

Credit Spreads 2000–2009



Source: FRED, Federal Reserve Bank of St. Louis, and British Bankers' Association.

Note: The TED spread is the difference between the 3-month LIBOR rate and the constant maturity 3-month Treasury bill rate. The Baa spread is the difference between the constant maturity Baa rate and the 10-year constant maturity Treasury bond rate.

Application: fixed income pricing

Common misconceptions

- ▶ NB: $r_f + \text{spread} \equiv YTM$, yield-to-maturity. The YTM is usually derived from the observation of the bond price.
- ▶ The spread is not the same as the risk premium! Even in a world where all investors are risk neutral or the default of the bond is not correlated with the SDF, the spread will be positive if default is possible and entails a loss.
- ▶ A common mistake is to derive the physical probability π of default from equation (10), while this equation gives the risk neutral probability π^* of default! To the extent that default tends to occur in “bad” states of the world (with low aggregate wealth), we expect that $\pi^* > \pi$.
- ▶ This mistake, which does not take into account risk aversion, leads to the overestimation of the physical proba of default.
 - ▶ It was very common in 2008, with people claiming that the probabilities of default implied by observed bond prices were implausibly high.

- ▶ Acknowledgements: Some sources for this series of slides include:
 - ▶ The slides of Martin Boyer, for the same course at HEC Montreal.
 - ▶ *Asset Pricing*, by John H. Cochrane.
 - ▶ *Finance and the Economics of Uncertainty*, by Gabrielle Demange and Guy Laroque.
 - ▶ *The Economics of Risk and Time*, by Christian Gollier.