The stochastic discount factor and the CAPM

Pierre Chaigneau
pierre.chaigneau@hec.ca

November 8, 2011
▶ Can we price all assets by appropriately discounting their future cash flows?
▶ What determines the risk premium on any asset?
▶ What is the mean-variance frontier?
▶ How do we obtain the CAPM from first principles?
▶ Suppose all assets are expected to pay higher dividends in the future. Should their prices rise?
The consumer problem (1)

> Multifaceted problem: how much to save for next period (intertemporal saving problem), and asset allocation.

> Standard problem in macroeconomics. The solution is given by the Euler equation.

> The agent maximizes the discounted utility of this period ($t$) and next period ($t + 1$) consumption (denoted by $c$). He is endowed with $e_t$ this period and $e_{t+1}$ the next (think about labor income). Consider any asset with price $p_t$ and random payoff $\tilde{x}_{t+1}$. Denote by $\alpha$ the quantity of this asset purchased. We assume that utility is time-separable, with subjective discount factor $\delta$.

$$\max_{\alpha} u(c_t) + \delta E_t[u(c_{t+1})]$$

s.t. \hspace{5mm} c_t = e_t - \alpha p_t \hspace{5mm} \text{and} \hspace{5mm} c_{t+1} = e_{t+1} + \alpha x_{t+1}$$
The consumer problem (2)

- Plug $c_t$ and $c_{t+1}$ into the objective function and take the FOC to get:

$$p_t u'(c_t) = E_t[\delta u'(c_{t+1})x_{t+1}] \tag{1}$$

$$p_t = E_t[\delta \frac{u'(c_{t+1})}{u'(c_t)}x_{t+1}] \tag{2}$$

- Interpretation of (1): equate marginal loss and marginal gain at the margin, in utility terms and in present values.
The stochastic discount factor (asset prices)
The basic asset pricing formula

- We define the *stochastic discount factor* \( m_{t+1} \) as

\[
m_{t+1} \equiv \delta \frac{u'(c_{t+1})}{u'(c_t)}
\]

- It is the marginal rate of substitution, i.e., the rate at which the investor is willing to substitute time \( t+1 \) consumption for time \( t \) consumption.

- We rewrite (2) as

\[
 p_t = E_t[m_{t+1}x_{t+1}]
\]
The stochastic discount factor (asset prices)

Corporate Finance

- A firm takes some (real/physical) investment decisions (capital budgeting), and raises money on the financial markets to fund these projects.

- What will be the equilibrium price (i.e., the maximum price that investors will be willing to pay) of a claim on a random sequence of future cash flows from a given set of projects?
  - Market value of equity of a firm.
  - How much money can be raised by issuing equity.

- It depends to what extent the cash flows generated by its projects covary with aggregate consumption.
  - It also depends on the risk preferences of investors. Time-varying risk aversion? What is the optimal economic context to fund a risky project? What about a risk-free project?
The stochastic discount factor (returns)

- For any future payoff, define the realized gross return as
  \[ R_{t+1} \equiv 1 + r_{t+1} \equiv \frac{x_{t+1}}{p_t}. \]
- Plugging into the SDF formula in (3):

  \[ 1 = E_t[m_{t+1}R_{t+1}] \quad (4) \]
The riskfree rate and the price of time

Consider the future payoff equal to $x_{t+1}$ with probability one (no uncertainty). We know that the present value of this payoff is

$$p_t = \frac{1}{1 + r_f} x_{t+1} \quad (5)$$

Applying (4) to the riskfree asset:

$$E[m] = \frac{1}{1 + r_f}$$

The average value of $m$ gives the value of time.

The riskfree rate exists and can be calculated, based on macroeconomic fundamentals ($c$), time preferences ($\delta$) and risk preferences ($u'$), even if there is no riskfree asset available.
The price of risk (asset prices)

\[ p = E[\text{mx}] = \text{cov}(m, x) + E[m]E[x] \]  

\[ p = \frac{E[x]}{R_f} + \text{cov}(m, x) \]  

\[ p = \frac{E[x]}{R_f} + \delta \frac{\text{cov}(u'(c_{t+1}), x)}{u'(c_t)} \]

- The value of a payoff is the discounted expected value of the future payoff plus an adjustment for risk. Assets whose future payoffs positively covary with the SDF (i.e., with future marginal utility) are more valuable, and conversely.

- Compare purchasing stocks to purchasing insurance.
The price of risk (expected excess return)

\[ 1 = E[mR] = \text{cov}(m, R) + E[m]E[R] \] (7)

- Divide through by \( E[m] \), remember that \( R_f = \frac{1}{E[m]} \):

\[ R_f = \frac{\text{cov}(m, R)}{E[m]} + E[R] \]

\[ E[R] = R_f - \frac{\text{cov}(m, R)}{E[m]} \] (8)

\[ E[R] - R_f = -\frac{\text{cov}(u'(c_{t+1}), R)}{E[u'(c_{t+1})]} \]

- An asset whose payoff covaries positively with consumption (negatively with the marginal utility of consumption) will demand a higher expected return, or equivalently a lower price.
Idiosyncratic risk (1)

- The equations above show that if

$$cov(m, x) = 0$$

then

$$p = \frac{E[x]}{R_f}$$

$$cov(m, R) = 0$$

then

$$E[R] = R_f$$

- If the payoff (or equivalently the return) of an asset is uncorrelated with the SDF, then the risk premium is zero, its expected return is equal to the riskfree rate, and the asset price is equal to the expected payoff discounted at the riskfree rate.

- Idiosyncratic risk does not matter in asset pricing.
Decompose any payoff $\tilde{x}$ into the part that is correlated with the SDF, which we call the \textit{systematic component}, and the part that is not, which we call the \textit{idiosyncratic component}:

$$x = \text{proj}(x|m) + \epsilon$$

$$\text{proj}(x|m) \equiv \frac{E[mx]}{E[m^2]} m$$

$$p(\text{proj}(x|m)) = p\left(\frac{E[mx]}{E[m^2]} m\right) = E\left[\frac{E[mx]}{E[m^2]} m^2\right] = E[mx] = p(x)$$
The asset’s beta

- Rewrite (8) as

\[ E[R_i] = R_f + \frac{\text{cov}(m, R_i)}{\text{var}[m]} \left( - \frac{\text{var}[m]}{E[m]} \right) \] (9)

\[ E[R_i] = R_f + \beta_{i,m} PR \] (10)

- The price of risk: \( PR \) is the same for all assets (independent of \( i \)).
- The quantity of risk in each asset: \( \beta_{i,m} \).
- This decomposition resembles the CAPM decomposition, but it is different.\(^1\)

\(^1\)In the CAPM, the price of risk is the equity premium (if the only risky assets available are stocks), while \( \beta \) is defined as the value of the coefficient in a univariate regression of asset \( i \) return on stock market returns.
The mean-variance frontier

Derived without the need to assume mean-variance preferences!

\[
1 = E[mR_i] = E[m]E[R_i] + \rho(m, R_i)\sigma_m\sigma_{R_i}
\]  

\[
E[R_i] = R_f - \rho(m, R_i)\frac{\sigma_m}{E[m]}\sigma_{R_i}
\]  

\[
|E[R_i] - R_f| \leq \frac{\sigma_m}{E[m]}\sigma_{R_i}
\]  

- The set of possible mean and variance of returns is bounded.
- The boundary is called the **mean-variance frontier**.
The mean-variance frontier: idiosyncratic risk

- Assets which lie on the frontier are perfectly correlated with the SDF, i.e., their idiosyncratic risk is nil.
- The (horizontal) distance to the frontier in the \( \{\sigma_{R_i}, E[R_i]\} \) plane is a measure of idiosyncratic risk: intuitively, this part of the standard deviation of returns does not generate a compensation in the form of higher expected return.
The mean-variance frontier

- For any asset, equation (12) may be rewritten as

\[ E[R_i] = R_f - \frac{\text{cov}(m, R_i)}{E[m]} \]  

(14)

- Besides, given a point \( k \) on the mean-variance frontier which is perfectly negatively correlated with the SDF \( (\rho(m, R_k) = -1) \) and satisfies \( \sigma_{R_k} = \sigma_m \). Then

\[ \frac{1}{E[m]} = \frac{E[R_k] - R_f}{\sigma_m^2} \]  

(15)

- Plugging in (14):

\[ E[R_i] = R_f - \beta_i(m, R_i)[E[R_k] - R_f] \]  

(16)

\[ E[R_i] = R_f + \beta_i(R_k, R_i)[E[R_k] - R_f] \]  

(17)
The CAPM

- The SDF involves consumption. Can we express the expected returns on any asset as a function of the expected returns on another asset or portfolio of assets?
- This is the principle of the CAPM. It can be derived with quadratic utility, or normally distributed asset returns.
  - In any of these two cases, the expected utility associated with any portfolio (a portfolio is a combination of assets) is fully described by the mean and the variance of returns of this portfolio.
  - In this case, any investor will select a portfolio among a set of mean-variance efficient portfolios.
- The set of mean-variance efficient portfolios is the set of portfolios with the maximum expected return for a given variance of return – or equivalently the set of portfolios with the minimum variance of returns for a given expected return.
The CAPM

- If there are $J$ assets, the optimal asset weights $\{\omega_1, \ldots, \omega_J\}$ in the investor’s portfolio are given by

$$\min_{\{\omega_1, \ldots, \omega_J\}} \sum_{j=1}^{J} \sum_{i=1}^{J} \omega_j \omega_i \text{cov}(r_j, r_i)$$

subject to

$$\sum_{j=1}^{J} \omega_j E[r_j] = E[r] \quad \text{and} \quad \sum_{j=1}^{J} \omega_j = 1$$

where $r_j$ is the return on asset $j$. Note that $\text{cov}(r_j, r_j) = \text{var}(r_j)$, and $E[r]$ is the given required expected return on the portfolio.

- Denoting the Lagrange multipliers associated to the first and second constraint by $\lambda$ and $\mu$ respectively, the first-order conditions to this optimization problem are

$$2 \sum_{j=1}^{J} \omega_j \text{cov}(r_i, r_j) - \lambda E[r_i] - \mu = 0 \quad \text{for } i = 1, \ldots, J$$
The CAPM

Denote by $r_m = \sum_{j=1}^{J} \omega_j r_j$ the return on the market portfolio. Then

$$cov(r_m, r_i) = cov\left( \sum_{j=1}^{J} \omega_j r_j, r_i \right) = \sum_{j=1}^{J} \omega_j cov(r_j, r_i)$$

Using this equation, we can rewrite the FOC as

$$2cov(r_m, r_i) - \lambda E[r_i] - \mu = 0 \quad \text{for } i = 1, \ldots, J \quad (18)$$

This equation should hold for the market portfolio:

$$2var[r_m] - \lambda E[r_m] - \mu = 0$$

This equation should also hold for the riskfree asset:

$$-\lambda r_f - \mu = 0$$
The CAPM

- Combining these two equations:
  \[ \lambda = \frac{2 \text{var}[r_m]}{E[r_m] - r_f} \quad \text{and} \quad \mu = -\frac{2 \text{var}[r_m]}{E[r_m] - r_f} r_f \]

- Plugging these values of \( \lambda \) and \( \mu \) in (18),
  \[
  2 \text{cov}(r_m, r_i) - \frac{2 \text{var}[r_m]}{E[r_m] - r_f} E[r_i] + \frac{2 \text{var}[r_m]}{E[r_m] - r_f} r_f = 0
  \]
  Or
  \[
  E[r_i] - r_f = \frac{\text{cov}(r_i, r_m)}{\text{var}[r_m]} [E[r_m] - r_f]
  \]
The CAPM

- We define

\[ \beta_i \equiv \frac{\text{cov}(r_i, r_m)}{\text{var}[r_m]} \]

- The beta of any asset measures its contribution to the riskiness of the market portfolio. Because of our assumptions, the riskiness is measured by the variance of returns.

- The expected excess return on any asset \( i \) is

\[ E[r_i] - r_f = \beta_i [E[r_m] - r_f] \]

- The right-hand-side of this equation is the risk premium. With the CAPM, the risk premium is given by the quantity of risk in asset \( i \), as measured with \( \beta_i \), multiplied by the price of risk, as measured by the expected excess return on the market portfolio of risky assets.
In practice: correlations in 2010-2011

Important caveat: correlations change over time!

- T-bonds are negative beta assets over this time period.
- Source: http://www.assetcorrelation.com/user/correlations/366

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The discount rate effect and the payoff effect
With log utility

- The wealth portfolio is a claim to all future consumption. Intuitively, if one owns all (respectively 1%) of the assets of all types in the economy, he has a claim to 100% (resp. 1%) of all future consumption.

- The price of the wealth portfolio is

\[ p_t^M = E_t \left[ \delta \frac{u'(c_{t+1})}{u'(c_t)} c_{t+1} \right] \]

- With log utility \( u(c) = \ln(c) \):

\[ p_t^M = \delta c_t \]
The discount rate effect and the payoff effect

▶ Strangely, the portfolio price $p^M_t$ does not depend on future consumption.

▶ This is because two effects exactly offset each other with log utility. On the one hand, higher future consumption raises $p^M_t$ (direct payoff effect). On the other hand, higher future consumption diminishes the SDF, which decreases $p^M_t$ (indirect discounting effect).

▶ Importantly, this only holds for the wealth portfolio. This would *not* be verified for individual assets: any news of higher future payoff would have a negligible effect on the SDF, so that the payoff effect would be dominant.

▶ This effect must be kept in mind when there are news that the economy is becoming more productive (think about the late 1990s). Should this raise asset prices? Not necessarily!
Acknowledgements: Some sources for this series of slides include:

- The slides of Martin Boyer, for the same course at HEC Montreal.
- *The Economics of Risk and Time*, by Christian Gollier.
The CAPM
As derived with CRRA utility

- The SDF involves consumption. Can we express the expected returns of any asset as a function of the expected returns of another asset or portfolio of assets?
- This is the principle of the CAPM. It can be derived with quadratic utility or normally distributed returns.
- Here, we will derive it with CRRA utility, which is a commonly used utility function in finance, and we won’t assume anything on the distribution of returns. However, in discrete time, we will need to work with approximations.
- The *market portfolio* includes all existing assets in the economy (stocks, bonds, real estate, commodities, etc.). Cochrane talks about a wealth portfolio which is a claim to all future consumption.
The CAPM
As derived with CRRA utility

- Denote by $R_{t+1}^M$ the return on the wealth portfolio.
- For agents with infinite horizons, $\frac{c_{t+1}}{c_t} = \frac{W_{t+1}}{W_t} = R_{t+1}^M$.
- It follows that $m = \delta \left( \frac{c_{t+1}}{c_t} \right)^{-\gamma} = \delta (R_{t+1}^M)^{-\gamma}$.
- Use a first order Taylor expansion to rewrite this term as
  \[
  \delta (R_M)^{-\gamma} \approx \delta (\overline{R_M})^{-\gamma} + \delta (-\gamma) (\overline{R_M})^{-\gamma-1} (R_M - \overline{R_M})
  \]
The CAPM

As derived with CRRA utility

- Equation (14) rewrites as

\[ E[R_i] - R_f = -\frac{\text{cov}(m, R_i)}{E[m]} \]

\[ E[R_i] - R_f = -\frac{\text{cov}(-\delta\gamma(R_M)^{-\gamma-1}(R_M - \bar{R}_M), R_i)}{E[m]} \]

- For small time intervals, \( E[m] = R_f^{-1} \approx 1 \) and \( \delta(R_M)^{-\gamma-1} \approx 1 \)

\[ E[R_i] - R_f = \gamma \text{cov}(R_M, R_i) = \frac{\gamma \text{cov}(R_M, R_i)}{\text{var}(R_M)} \text{var}(R_M) \]

\[ E[R_i] - R_f = \gamma \beta(R_i, R_M) \text{var}(R_M) \quad (19) \]
The CAPM
As derived with CRRA utility

- Setting $i = M$ in (19) gives:
  \[ \gamma \text{var}(R_M) = E[R_M] - R_f \]

- Substituting in (19):
  \[ E[R_i] - R_f = \beta_i [E[R_M] - R_f] \]

- Remember that we used approximations to reach this result.
The CAPM
As derived with CRRA utility and consumption instead of portfolio returns

▶ Use a first order Taylor expansion to rewrite the SDF as

\[ m = \delta \left( \frac{c_{t+1}}{c_t} \right)^{-\gamma} \approx \delta \left( \frac{c_{t+1}}{c_t} \right)^{-\gamma} - \gamma \delta \left( \frac{c_{t+1}}{c_t} \right)^{-\gamma-1} \left( \frac{c_{t+1}}{c_t} - \frac{c_{t+1}}{c_t} \right) \]

▶ On a short time interval, \( E(m) = \frac{1}{R_f} \approx 1 \) and

\[ \delta \left( \frac{c_{t+1}}{c_t} \right)^{-\gamma-1} \approx 1. \text{ Finally,} \]

\[ m \approx 1 - \gamma \left( \frac{c_{t+1}}{c_t} - \frac{c_{t+1}}{c_t} \right) \]
The CAPM

As derived with CRRA utility and consumption instead of portfolio returns

- Equation (14) rewrites as

\[
E[R_i] - R_f = -\frac{\text{cov}(m, R_i)}{E[m]}
\]

\[
E[R_i] - R_f \approx -\text{cov}(-\gamma\left(\frac{c_{t+1}}{c_t} - \frac{c_{t+1}}{c_t}\right), R_i)
\]

\[
E[R_i] - R_f \approx \gamma \text{cov}\left(\frac{c_{t+1}}{c_t}, R_i\right)
\]

\[
E[R_i] - R_f \approx \gamma \frac{\text{cov}\left(\frac{c_{t+1}}{c_t}, R_i\right)}{\text{var}\left(\frac{c_{t+1}}{c_t}\right)} \text{var}\left(\frac{c_{t+1}}{c_t}\right)
\]

\[
E[R_i] - R_f \approx \gamma \beta(\Delta c, R_i) \text{var}\left(\frac{c_{t+1}}{c_t}\right)
\]
The CAPM

Some lessons, and some perspective

- $\gamma \text{var} \left( \frac{c_{t+1}}{c_t} \right)$ is the price of risk in the economy.

- It involves risk aversion and the variability of aggregate consumption: risk preferences and macroeconomic fundamentals.

- The intuition as to why risk can be measured by the variance of consumption (or returns) in continuous time is because preferences are locally (for small variations) mean-variance.
  - Continuous time models often assume a Brownian motion process, which is continuous: for a very small time interval, stock price variations are “small”.
  - Individuals are assumed to make portfolio choice decisions for an arbitrarily small time interval, and re-optimize continuously.

- Jumps introduce a discontinuity in the stock price process. In this case, even in continuous time and continuous portfolio choice, preferences are not mean-variance.
More on the wealth portfolio

- With log utility, the price of the wealth portfolio with an infinity of future periods (instead of one) is

\[
p_t^M = E_t \left[ \sum_{\tau=1}^{\infty} \delta^\tau \frac{u'(c_{t+\tau})}{u'(c_t)} c_{t+\tau} \right] = E_t \left[ \sum_{\tau=1}^{\infty} \delta^\tau \frac{c_t}{c_{t+\tau}} c_{t+\tau} \right] = c_t \frac{\delta}{1 - \delta}
\]

- The return on the wealth portfolio is

\[
R_{t+1}^M \equiv \frac{p_{t+1}^M + c_{t+1}}{p_t^M} = \frac{\delta}{1 - \delta} + \frac{1}{c_t} c_{t+1}
\]

\[
= \frac{1}{\delta} \frac{c_{t+1}}{c_t} = \left( \delta \frac{u'(c_{t+1})}{u'(c_t)} \right)^{-1} = m_{t+1}^{-1}
\]